Limits, Continuity and derivatives (Stewart Ch. 2)

$\lim\limits_{x\to a} f(x) = L$

say: "the limit of *f(x)* equals *L* as *x* approaches *a*"

The values of *f(x)* can be as close to *L* as we like by taking *x* sufficiently close to *a*, but $x \neq a$.

If there is such a *L*, the limit $\lim_{x\to a} f(x)$ **<u>exists</u>**

One sided limits:

$\lim\limits_{x\downarrow a} f(x) = L$	"The right limit of *f(x)* equals *L* as *x* approaches *a* from the right"
$\lim\limits_{x\uparrow a} f(x) = L$	"The left limit of *f(x)* equals *L* as *x* approaches *a* from the left"

Notation: $x \uparrow a \;\Leftrightarrow x \to a^-$ and $x \downarrow a \;\Leftrightarrow x \to a^+$

<u>Theorem:</u>

$$\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\downarrow a} f(x) = L \text{ and } \lim_{x\uparrow a} f(x) = L$$

Infinite limits:

$\lim\limits_{x\longrightarrow a} f(x) = \infty$	"*f(x)* becomes infinite as *x* approaches *a*"
$\lim\limits_{x\longrightarrow a} f(x) = -\infty$	"*f(x)* becomes negative infinite as *x* approaches *a*"

<u>Note 1</u>: ∞ and $-\infty$ are not real numbers!

<u>Note 2</u>: similar expressions for infinite one sided limits

<u>Note 3</u>: if *f(x)* becomes ∞ or $-\infty$ for *x* approaches *a*,

x* = *a is a **vertical asymptote** of the graph *y* = *f(x)*

The graph $y = f(x)$ has a **horizontal asymptote** $\boldsymbol{y = a}$ if

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

Limits at infinity: "$f(x)$ can be made arbitrarily close to L as x approaches ∞ or $-\infty$"

Limit Laws: if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist

$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
$\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$
$\lim_{x\to a}[f(x)\,g(x)] = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$
$\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
$\lim_{x\to a} c = c$, $\lim_{x\to a} x = a$ and $\lim_{x\to a} x^n = a^n$
$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ and $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, if $\sqrt[n]{a}$ and $\sqrt[n]{\lim_{x\to a} f(x)}$ exist

Direct substitution rule for a polynomial or rational function $f(x)$: $x \in \text{Domain}_f \Rightarrow \lim_{x\to a} f(x) = f(a)$

If $f(x) = g(x)$ for all $x \neq a$ then:

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) \quad \text{if the limits exist.}$$

If $f(x) \leq g(x)$ when x is near a then:

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x) \quad \text{if the limits exist.}$$

The Squeeze Theorem:

If $f(x) \leq g(x) \leq h(x)$ when x is near a then:

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = \mathrm{L} \quad \Rightarrow \quad \boldsymbol{\lim_{x\to a} g(x) = \mathrm{L}}$$

<u>Continuity of functions:</u>

f is **continuous at a** if $\boldsymbol{\lim_{x\to a} f(x) = f(a)}$

This means: **1.** $f(x)$ is at a , **2.** $\lim_{x\to a} f(x)$ exists and

3. $\lim_{x\to a} f(x)$ equals $f(a)$.

Discontinuity of f at a is

- **infinite discontinuity**: $x = a$ is a vertical asymptote
- **jump discontinuity**: $\lim_{x\downarrow a} f(x) \neq \lim_{x\uparrow a} f(x)$
- **removable discontinuity:**
  $\lim_{x\downarrow a} f(x) = \lim_{x\uparrow a} f(x) = \mathrm{L}$, so that f can be (re)defined in a: $f(a) = L$.

f is **continuous from the left** if $\lim_{x\uparrow a} f(x) = f(a)$

and **continuous from the right** if $\lim_{x\downarrow a} f(x) = f(a)$

f is **continuous on an interval** ⇔

f is continuous in every number of the interval.

(If the interval is closed like $[a, b]$, f is continuous from the right in a and continuous from the left in b.

If f and g are continuous functions at a, then **combinations** of f and g are continuous at a:
cf, $f+g$, $f-g$, $f \times g$ and, if $g(a) \neq 0$, f/g

Functions, that are **continuous on their domain**: polynomial, rational, root, trigonometric, exponential, logarithmic functions and all inverse functions of continuous functions.

Continuity of a composite function $f \circ g$:
If g is continuous at a and f is *continuous* at $g(a)$, then $f \circ g\,(x) = f[g(x)]$ is continuous at a.

The Intermediate Value Theorem
If f is continuous on $[a, b]$ and N is an arbitrary number between $f(a)$ and $f(b)$, then there is a c such that $f(c) = N$.

Some special limits:

$\lim_{x\to\infty} tan^{-1}(x) = \frac{1}{2}\pi$ and $\lim_{x\to\infty} tan^{-1}(x) = \frac{1}{2}\pi$	
If $a > 0$, then $\mathbf{lim}_{x\to\infty} \frac{\mathbf{1}}{x^{a}} = \mathbf{0}$	And if $a > 0$ and x^a is de ined for $x < 0$, then $\lim_{x\to-\infty} \frac{\mathbf{1}}{x^{a}} = \mathbf{0}$

$\lim_{x\to\infty} e^{x} = \infty$ means: "e^x can be made arbitrarily large as x is suf iciently large" .

<u>Derivatives:</u> the **tangent line** of the curve $y = f(x)$ at point $(a, f(a))$ has **slope** $\boldsymbol{m = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}}$

Or, substituting $x = a+h$: $\boldsymbol{m = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}}$

This is **the derivative *f `(a)*** of f at a:

$$\boldsymbol{f`(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}}$$

The derivative as a **rate of change:**

The **rate of change** of the graph $y = f(x)$ from point $(x_1, f(x_1))$ to $(x_2, f(x_2))$ is $\frac{\Delta y}{\Delta x} = \boldsymbol{\frac{f(x_2)-f(x_1)}{x_2-x_1}}$.

Then $\boldsymbol{f`(x_1) = \lim_{x_2 \to x_1} \frac{f(x_2)-f(x_1)}{x_2-x_1} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}}$

(the **instantaneous** rate of change of f at x_1)

f is **differentiable at *a*** if *f `(a)* exists and

f is **differentiable at *(a, b)*** if *f `(x)* exists for all $x \in (a, b)$: then *f `(x)* can be seen as a function on *(a, b)*.

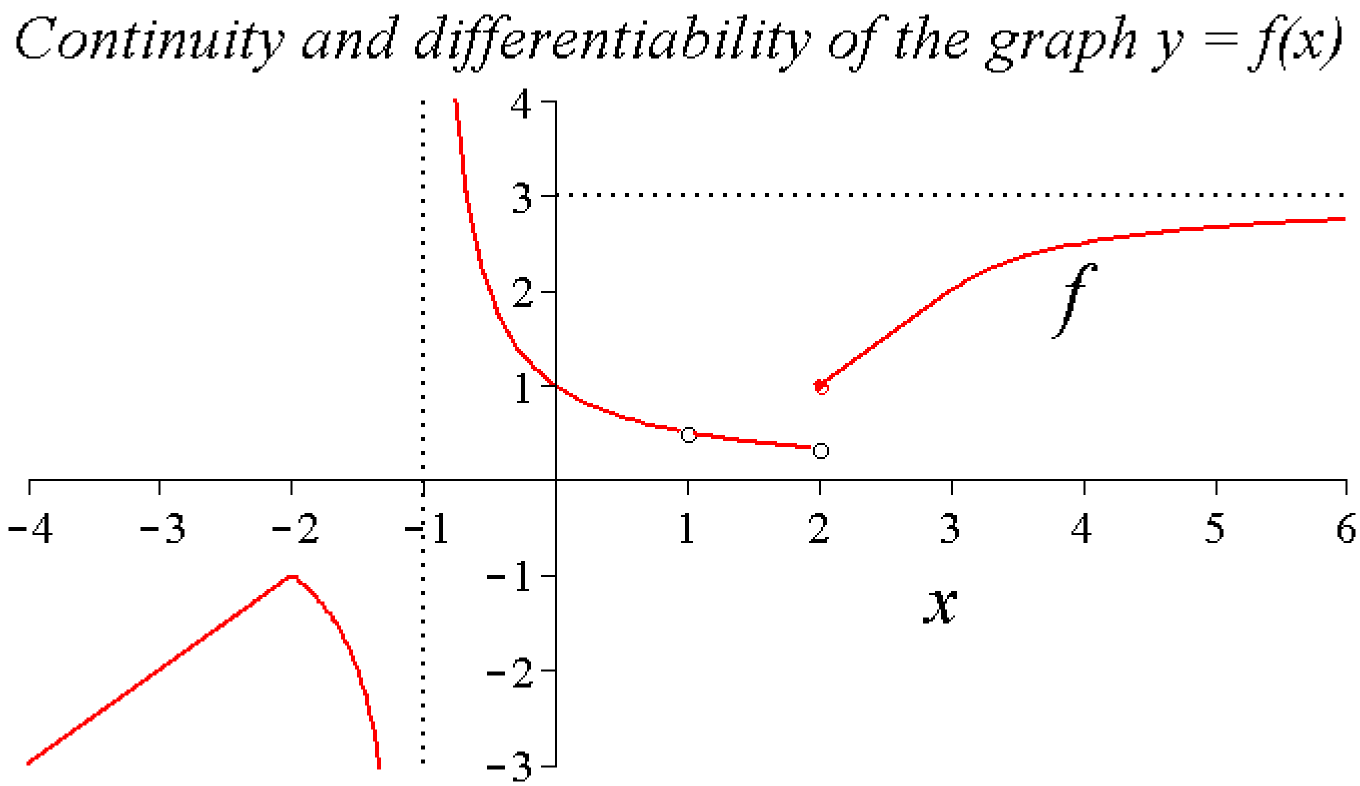
<u>Notations of a derivative of *f*</u>: $\boldsymbol{f`(x) = \frac{df}{dx} = \frac{dy}{dx}}$

f is differentiable at *a* => *f* is continuous at *a* (<u>not v.v</u>!)

Second derivative: $\boldsymbol{f``(x) = \frac{d}{dx} f`(x) = \frac{d^2f}{dx^2} = \frac{d^2y}{dx^2}}$

Continuity and differentiability by (1) example of a piecewise defined function:

$$f(x) = \begin{cases} x+1 & \text{if} \quad x < -2 \\ \dfrac{x-1}{x^2-1} & \text{if} \ -2 \le x < 2 \\ x-1 & \text{if} \quad 2 \le x < 3 \\ 3 + \dfrac{1}{x-2} & \text{if} \quad x \ge 3 \end{cases}$$



Domain of f : $D_f = \{x \in \mathbb{R} | x \neq -1, 1\}$

Range: $R_f = \left\{x \in \mathbb{R} \middle| x < -1 \text{ or } \frac{1}{3} < x < \frac{1}{2} \text{ or } x > \frac{1}{2}\right\}$

<u>Asymptotes:</u>

$$\left.\begin{aligned}\lim_{x\uparrow -1} f(x) &= \lim_{x\uparrow -1} \frac{x-1}{(x-1)(x+1)} = -\infty \\ \lim_{x\downarrow -1} f(x) &= \lim_{x\downarrow -1} \frac{x-1}{(x-1)(x+1)} = +\infty\end{aligned}\right\}$$

$\Rightarrow$ $\boldsymbol{x = -1}$ is a **vertical aymptote**

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(x+1) = -\infty$ (does not exist!)

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(3 + \frac{1}{x-2}\right) = 3 \Rightarrow$

$\boldsymbol{y = 3}$ is a **horizontal asymptote**

Limits in other potential points of discontinuity:

x = -2

$$\left.\begin{aligned}\lim_{x\uparrow -2} f(x) &= \lim_{x\uparrow -2} (x+1) = -1 \\ \lim_{x\downarrow -2} f(x) &= \lim_{x\downarrow -2} \frac{x-1}{x^2-1} = -1\end{aligned}\right\} \Rightarrow \boldsymbol{\lim_{x\to-2} f(x) = -1}$$

x = 1:

$$\boldsymbol{\lim_{x\to1} f(x)} = \lim_{x\to1} \frac{x-1}{(x-1)(x+1)} = \lim_{x\to1} \frac{1}{x+1} = \boldsymbol{\frac{1}{2}}$$

x = 2:

$$\left.\begin{aligned}\lim_{x\uparrow 2} f(x) &= \lim_{x\uparrow 2} \frac{x-1}{x^2-1} = \frac{1}{3} \\ \lim_{x\downarrow 2} f(x) &= \lim_{x\downarrow 2}(x-1) = 1\end{aligned}\right\} \Rightarrow \boldsymbol{\lim_{x\to-2} f(x)} \textbf{ does not exist}$$

$x = 3$:

$$\left.\begin{aligned}\lim_{x\uparrow 3} f(x) &= \lim_{x\uparrow 3}(x-1) = 2\\ \lim_{x\downarrow 3} f(x) &= \lim_{x\downarrow 3}\left(3-\frac{1}{x-2}\right) = 2\end{aligned}\right\} \Rightarrow \boldsymbol{\lim_{x\to 3} f(x) = 2}$$

a	**Continuous at *a* ?** $\lim_{x\to a} f(x) = f(a)$	**Differentiable at *a*?** $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists
-2	**Yes**: $\lim_{x\to -2} f(x) = -1 = f(-2)$	**No:** $\lim_{h\uparrow 0}\frac{f(-2+h)-f(-2)}{h} \neq \lim_{h\downarrow 0}\frac{f(-2+h)-f(-2)}{h}$
-1	**No**: asymptotic discontinuity	**No**: *f* is not differentiable, if *f* is not continuous
1	**No**, but removable discontinuity: define *f*(1)= ½	**No**: not continuous
2	**No**: jump discontinuity	**No**: not continuous
3	**Yes:** $\lim_{x\to 3} f(x) = 2 = f(3)$	**Yes:** $\lim_{h\uparrow 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\downarrow 0}\frac{f(3+h)-f(3)}{h}$

If *a* ≠ *-2, -1, 1, 2, 3*	*f* is **continuous and differentiable at *a*** , because then *f(x)* is a polynomial or rational function: these are always continuous and differentiable on their domain.