<u>Derivatives: rules and applications</u> (Stewart Ch. 3/4)

The derivative $f'(x)$ of the function *f(x)*:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(for all *x* for which *f* is differentiable/ the limit exists)

<u>Property</u>:if *f* is differentiable at *x*, then *f* continuous at *x*

("continuity is a condition for differentiability")

$f(x)$	x	x^2	x^3	$\frac{1}{x} = x^{-1}$	$\frac{1}{x^2} = x^{-2}$	$\sqrt{x} = x^{\frac{1}{2}}$
$f'(x)$	1	$2x$	$3x^2$	$-x^{-2} = -\frac{1}{x^2}$	$-2x^{-3} = -\frac{2}{x^3}$	$\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

Power rule (derivatives of powers of x , $n \neq 0$):

$$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$$

Differentiation rules:

Function	Derivative
$f(x) = c$	$f'(x) = 0$
$cf(x)$	$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$
$f(x) + g(x)$ $f(x) - g(x)$	$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$ $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$
$f(x)g(x)$	$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Short notations: $(cf)' = cf'$ and $(f \pm g)' = f' \pm g'$

Product rule: $(fg)' = f'g + fg'$

Quotient rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Derivatives of common functions:

$f(x) = c$	$\Rightarrow$	$f'(x) = 0$
$f(x) = x^n$	$\Rightarrow$	$f'(x) = nx^{n-1}$
$f(x) = e^x$	$\Rightarrow$	$f'(x) = e^x$
$f(x) = \ln(x)$	$\Rightarrow$	$f'(x) = \frac{1}{x}$
$f(x) = \sin(x)$	$\Rightarrow$	$f'(x) = \cos(x)$
$f(x) = \cos(x)$	$\Rightarrow$	$f'(x) = -\sin(x)$
$f(x) = \tan(x)$	$\Rightarrow$	$f'(x) = \frac{1}{\cos^2(x)}$
$f(x) = a^x$	$\Rightarrow$	$f'(x) = a^x \ln(a)$
$f(x) = \log_a(x)$	$\Rightarrow$	$f'(x) = \frac{1}{x \ln(a)}$

The Chain Rule for $f \circ g(x) = f(g(x))$

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

Or, in the Leibnitz notation for $u = g(x)$ and $v = f(u)$:

$$\frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx}$$

Applications of the chain rule:

1. $\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$

2. $\frac{d}{dx}a^x = \frac{d}{dx}e^{\ln a^x} = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\ln a$

So, if $f(x) = a^x$, then $f'(x) = a^x \ln(a)$

<u>**Maxima and minima**</u>: extreme values $f(c)$ of f at c:

- $f(c)$ is a **absolute** maximum if $f(x) \leq f(c)$ for all x in D_f
- $f(c)$ is a **absolute** minimum if $f(x) \geq f(c)$ for all x in D_f
- The maximum or minimum is **local** if the above is true for "x near c".

<u>**The Extreme Value Theorem:**</u>

If $f(x)$ is **continuous at a closed interval** $[a, b]$, then f attains an absolute maximum and absolute minimum at $[a, b]$

<u>**Format's Theorem:**</u>

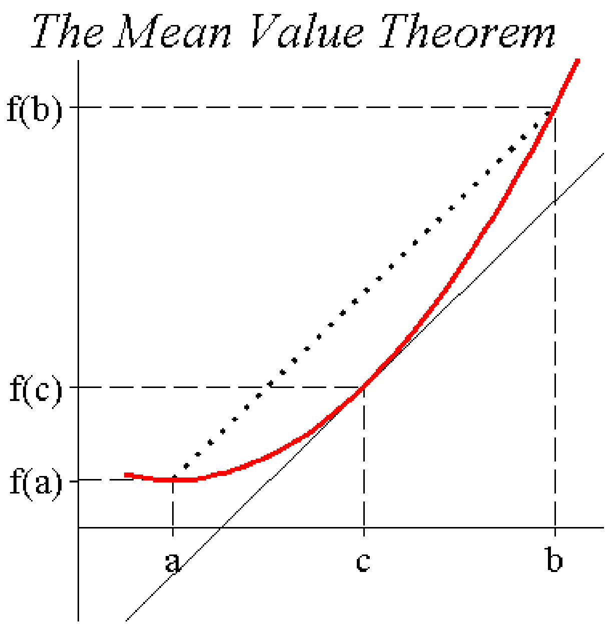
If f has a **local maximum or minimum at** c and f is differentiable at c, then $\boldsymbol{f'(c) = 0}$.

Definition: c is called a **critical value** of a function f if $f'(c) = 0$ or $f'(c)$ does not exist.

The closed interval method: finding extreme values of a continuous function on a closed interval $[a, b]$:

1. Compute $f(c)$ for all critical numbers c of f.
2. Compute $f(a)$ and $f(b)$.
3. The largest and the smallest of the function values at 1. and 2. are the absolute maximum and minimum

The Mean Value Theorem:



If
1. f is continuous on $[a, b]$
2. f is differentiable on (a, b)
then:
there is a c in (a, b) such that

$$\boldsymbol{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

The increasing/decreasing test:

If for all x in an interval I:

- $\boldsymbol{f'(x) > 0}$ => f is an **increasing** function on I
- $\boldsymbol{f'(x) < 0}$ => f is a **decreasing** function on I

The first derivative test:

If, for a critical value c, $\boldsymbol{f'(c) = 0}$ and f' changes from positive to negative at c, then $f(c)$ is a local maximum, and $f(c)$ is a local minimum, if f'changes from positive to negative at c.

The concavity test If for x on an interval I:

- $\boldsymbol{f''(x) > 0}$ => f is **concave upward** on I
- $\boldsymbol{f''(x) < 0}$ => f is **concave downward** on I

The second derivative test: for continuous f'' near c

- If $f'(c) = 0$ and $f''(c) < 0$ => f has a local maximum at c.
- If $f'(c) = 0$ and $f''(c) > 0$ => f has a local minimum at c.

L' Hopital's Rule

for solving **indeterminate limits** (form $\frac{0}{0}$ or $\frac{\infty}{\infty}$):

If $f(x)$ and $g(x)$ are differentiable at a, then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

- This rule is also valid for limits at infinity ($x \longrightarrow \pm\infty$)
- Solving limits of form $0\times\infty$, using L' Hopital's Rule:

$$\lim_{x \to a} f(x)\, g(x) = \lim_{x \to a} \frac{f(x)}{1/g(x)} \quad \text{(now of form } \frac{0}{0}\text{)}$$

<u>Curve Sketching</u>

An extensive investigation of the curve of f covers:

1. The **domain** of f , D_f

2. The ***x*-intercepts**: roots of the equation $f(x) = 0$,
And the ***y*-intercept** $f(0)$.

3. Is f **even**: $f(-x) = f(x)$
odd: $f(-x) = -f(x)$ or
periodic?

4. Find the equations of the asymptotes:

Vertical asymptotes $x = a$: if $f(a)$ has form $\frac{c\neq 0}{0}$

Horizontal asymptotes $y = L$: $\lim_{x\to\pm\infty} f(x) = L$

5. Derivative $f'(x)$

- Solve $f'(x) = 0$ and give all critical values.
- Use the sign scheme of $f'(x)$ or the **second** derivative test to find all **local and absolute minima and maxima**

An application of the procedure above is **optimization of a function *f*:** find all minima and maxima of f on its domain.

Methods of limit solving (a summary):

1. Always try first **substitution: the substitution rule** can be applied to every function that is continuous on its domain.

2. If substitution leads to the form $\frac{c\neq 0}{0}$**,** we have a vertical asymptote: find the left and the right hand limit, by **reasoning (**∞ or -∞)

If substitution leads to the form $\boxed{\frac{0}{0} \text{ or } \frac{\infty}{\infty} \text{ or } 0\times\infty,}$ try:

3. Factorizing, especially for rational functions

4. The root trick for function with roots.

5. Limits at infinity of rational functions: **divide by the largest power of *x* in the denominator**. (for quotients of exponential functions: divide by the highest power of e^x in the denominator).

6. L'Hopital's Rule.

For special cases such as functions of cos(*x*) and sin(*x*):

7. The Squeeze Theorem.

The topics above are all part of the final test of Calculus I in June 2009

The following topics are <u>not</u> part of the final test - June 2009:

Inflection points of the function $f(x)$: points c where the graph of f changes from upward to downward concavity, or reverse:

These points can be found by:
1. Solving $f''(x) = 0$
2. Making a sign scheme of $f''(x)$: if sign changes, it is an inflection point.

Implicit differentiation:
If the relation of y and x is given by an equation (not by a function), find the derivative $y' = \frac{dy}{dx}$ by:

- Differentiating the equation with respect to x.
- Solve y' from the resulting equation.

Derivatives of the inverse trigonometric functions:

$$f(x) = \sin^{-1}(x) \quad \Rightarrow \quad f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f(x) = \cos^{-1}(x) \quad \Rightarrow \quad f'(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$f(x) = \tan^{-1}(x) \quad \Rightarrow \quad f'(x) = \frac{1}{1+x^2}$$